General Relativity I

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LIGO Livingston SURF Lecture

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General Relativity Lectures

I. Today (JTW): Special Relativity
Introduction to Spacetime Geometry
& Coördinate Transformations

II. July 23 (J. Romano): Curvature
Curved Geometry
& Motion of Particles in Curved Spacetime

III. July 31 (W. Anderson): Gravitation
Matter as the Origin of Spacetime Curvature

Outline of This Lecture

I. Defining Principles
   A. Relativity
   B. Invariance of Speed of Light

II. Spacetime Interval
   A. Invariance Replaces Invariance of Time & Distance
   B. Spacetime Diagrams

III. Geometry of Space and Spacetime
   A. Coördinate Systems & Transformations
   B. Space(time) Metric

Defining Principles of Einstein's Special Theory of Relativity

I. Relativity: Laws of physics the same for any inertial observer. (Inertial = constant velocity, i.e., not influenced by any force)

II. All inertial observers measure speed of light *in vacuo* as $c = 3.00 \times 10^8\,\mathrm{m/s}$

Principle of Relativity

Already applies to the physics of Galileo & Newton:

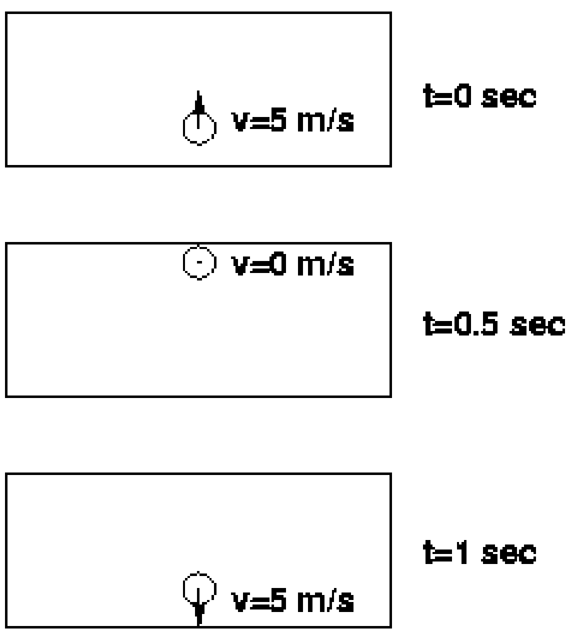
E.g., set up a lab (or play ping-pong or pool) in a moving train.

Observer on train & observer on the ground each set up Cartesian coördinate systems in which they are at rest.

Kinematics and dynamics in either coördinate system obey same laws of classical physics.

Example of Galilean Relativity

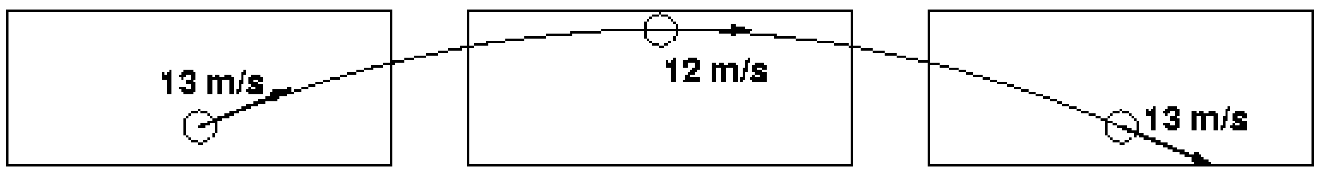
Toss ball in air in train moving at 12 m/s:
According to observer on the train,
ball goes straight up & comes straight down



t(s)	x(m)	y(m)	v_x(m/s)	v_y(m/s)
0	0	0	0	5
0.5	0	1.25	0	0
1	0	0	0	-5

Example of Galilean Relativity (cont'd)

According to observer on ground,
ball executes parabolic trajectory



t(s)	x(m)	y(m)	v_x(m/s)	v_y(m/s)
0	0	0	12	5
0.5	6	1.25	12	0
1	12	0	12	−5

Two observers measure different $x(t)$ (& generally $y(t)$),
but either one is consistent w/Newtonian gravity & mechanics

Comments on Galilean Relativity

- Doesn't apply in Aristotelean physics:
  For Aristotle, natural state of object is at rest
  & so ball released inside moving train will slow down
  as seen by outside observer & thus appear
  to observer on train to move backwards

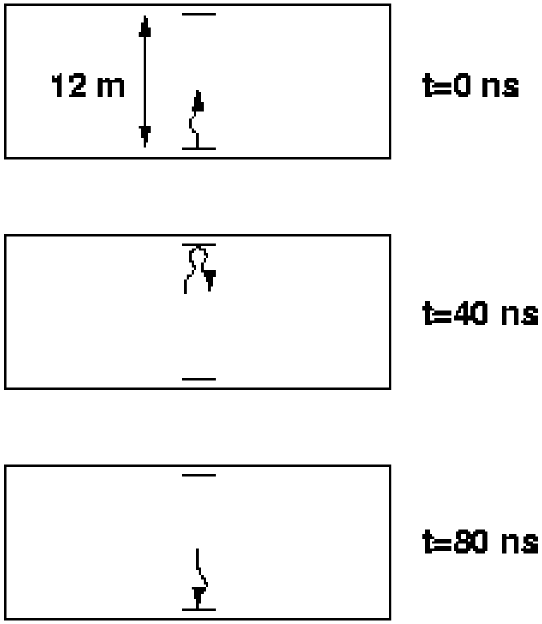
- Essential to make sense of physics on planet
  which is rotating at around 100 mph
  & orbiting the sun at around 4 million mph

Constancy of the Speed of Light

- Implied by Maxwell's Equations.
- Verified by Michelson-Morley Experiment (interferometer!)
- Inconsistent with Newtonian physics.

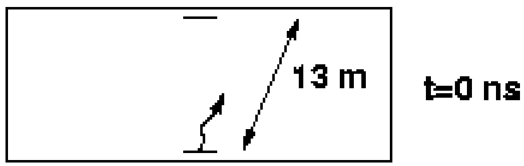
Speed of Light in Galilean Relativity

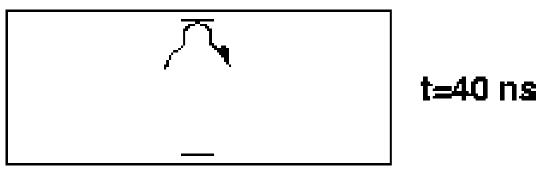
Bounce light between mirrors $12\,\text{m}$ apart in free-floating lab.
Light moves at $c = 3 \times 10^8\,\text{m/s} = 0.30\,\text{m/ns}$;
takes $40\,\text{ns}$ to travel between mirrors



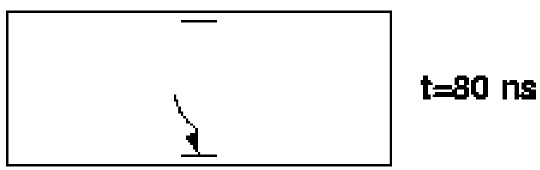
t(ns)	x(m)	y(m)
0	0	0
40	0	12
80	0	0

View same experiment while coasting past at $\frac{5}{12}c = 0.125\,\mathrm{m/ns}$;
In the $40\,\mathrm{ns}$ it takes light to travel between mirrors, lab moves $5\,\mathrm{m}$, so distance travelled is $13\,\mathrm{m}$, not $12\,\mathrm{m}$





$t(\mathrm{ns})$	$x(\mathrm{m})$	$y(\mathrm{m})$
0	0	0
40	5	12
80	10	0



Apparent speed of light is $\frac{13}{12}c = 0.325\,\mathrm{m/ns}$; what's wrong?

Inconsistency of Galilean Relativity w/Invariant Speed of Light

Sym arguments show both observers really do see $\Delta y = 12\,\mathrm{m}$, so light really does travel farther as seen by 2nd observer. Resolution is that this observer measures longer time btwn bounces. To make theory consistent w/invariance of speed of light, need to drop implicit invariance of time.

Describe experiment in more generality: two mirrors separated by a distance L (as measured by an observer at rest w.r.t. mirrors); work out coördinates of three **events** (successive bounces of the light) as measured by observer comoving w/mirrors (t', x', y') and one who sees them moving w/speed v in the positive x direction. (t, x, y)

Light Viewed in Two Reference Frames

Event	t	x	y	t'	x'	y'
A	$-\gamma L/c$	$-v\gamma L/c$	L	$-L/c$	0	L
B	0	0	0	0	0	0
C	$\gamma L/c$	$v\gamma L/c$	L	L/c	0	L

- Choose origin of both coördinate systems at middle bounce
- Distance L in both frames (symmetry)
- Mirrors stationary for primed observer
- Mirrors moving at speed v for unprimed observer
- Primed observer sees light moving at speed c

Unprimed observer also sees light moving at c:
From $\sqrt{(\Delta x)^2_{BC} + (\Delta y)^2_{BC}}/|(\Delta t)_{BC}| = c$, we find $\gamma = (1-v^2/c^2)^{-1/2}$.

Spacetime Interval

Consider the quantity

$$(\Delta s)^2 = -c^2(\Delta t)^2 + (\Delta \ell)^2$$

- By definition, $(\Delta s)^2 = 0 = (\Delta s')^2$ for two events connected by a light ray (e.g., AB, BC)

- But note that $(\Delta s)^2_{AC} = -4L^2 = (\Delta s')^2_{AC}$

- In fact, this is true in general. For any pair of events, the spacetime interval

$$(\Delta s)^2 = -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

is the same for all inertial observers.

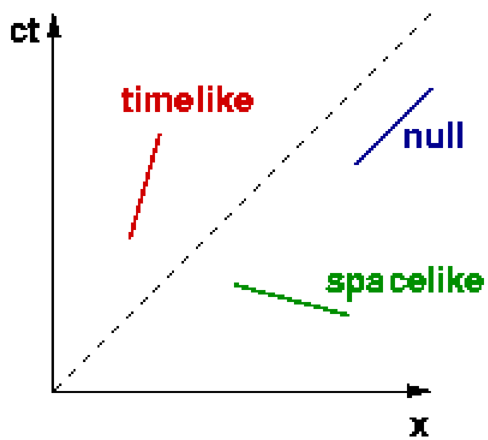
Meaning of the Interval

Consider three cases:

- $(\Delta s)^2 = 0$ "Null" or "Light-Like" interval:
  $|\Delta \ell|/|\Delta t| = c$ so events can lie along a light ray
- $(\Delta s)^2 < 0$ "Time-Like" interval:
  $|\Delta \ell|/|\Delta t| < c$ so events can lie along traj of sub-light inertial particle; in that ref frame, $\Delta \ell = 0$, so
  $\Delta \tau = \sqrt{-(\Delta s)^2/c^2} = \sqrt{(\Delta t)^2 - (\Delta \ell)^2/c^2} = \Delta t \rightarrow$ "proper time"
- $(\Delta s)^2 > 0$ "Space-Like" interval: $(\Delta s) = \sqrt{(\Delta \ell)^2 - c^2(\Delta t)^2}$
  $\equiv$ distance measured btwn events by inertial observer who sees them as occurring simultaneously $\longrightarrow$ "proper distance"

Spacetime Diagrams

For any observer, each event is located by its coörds (t, x, y, z); visualize relationships between events by plotting them on ct & x (& y & z) axes (use ct so light travels along 45° lines) Can use these to illustrate three types of intervals:



Intro to Spacetime Geometry

Make two notational simplifications:

- Work in units where $c = 1$ (defines what we mean by measuring time in meters and distance in (light-)seconds)

- Einstein summation convention: implied sum over repeated indices so for example
  $g_{\mu\nu}dx^\mu dx^\nu$ means $\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}dx^\mu dx^\nu$
  & $g_{ij}dx^i dx^j$ means $\sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij}dx^i dx^j$
  (where N is the number of space dimensions)

Spatial Geometry

For Euclidean flat space, the distance $\Delta\ell$ between two points is given by the Pythagorean theorem:

$$(\Delta\ell)^2 = (\Delta x)^2 + (\Delta y)^2 = \mathbf{\Delta x} \cdot \mathbf{\Delta x} = \delta_{ij}(\Delta x^i)(\Delta x^j)$$

where δ_{ij} is the Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Formula is unchanged if we change Cartesian coördinate systems

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Polar Coördinates

Change to coördinates (r, ϕ) such that $x = r\cos\phi$ and $y = r\sin\phi$. In general, no nice relationship btwn finite $(\Delta x, \Delta y)$ & $(\Delta r, \Delta\phi)$ but we can still look at infinitesimal (dx, dy) & $(dr, d\phi)$

Differentiating transformations gives

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix} \begin{pmatrix} dr \\ d\phi \end{pmatrix}$$

& substituting into infinitesimal distance gives

$$(d\ell)^2 = (dx)^2 + (dy)^2 = dr^2 + r^2 d\phi^2$$

& sure enough, this is also the geometrically-motivated distance formula

Spatial Metric

Infinitesimal distances measured with the metric

$$d\ell^2 = g_{k\ell}dx^k dx^\ell = g_{\overline{k\ell}}dx^{\overline{k}}dx^{\overline{\ell}}$$

The distance-squared $d\ell^2$ is a geometrical invariant, but the construction in different coörd systems may involve different components of the metric tensor g_{ij}, e.g.:

- $g_{xx} = g_{yy} = 1$; $g_{xy} = g_{yx} = 0$;
- $g_{\overline{xx}} = g_{\overline{yy}} = 1$; $g_{\overline{xy}} = g_{\overline{yx}} = 0$;
- $g_{rr} = 1$; $g_{\phi\phi} = r^2$; $g_{r\phi} = g_{\phi r} = 0$

Summary: Invariant $d\ell^2 = g_{ij}dx^i dx^j$; if $\{x^i\}$ is a Cartesian coörd system, $g_{ij} = \delta_{ij}$ (Euclidean metric) The metric describes infinitesimal distances & thus the geometry of space

Spacetime Metric

Recall the invariant spacetime interval:

$$(\Delta s)^2 = (\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$$

where $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$ and

$$\begin{aligned} \eta_{00} &= -1 \\ \eta_{11} &= \eta_{22} = \eta_{33} = 1 \\ \eta_{\mu\nu} &= 0, \qquad \mu \neq \nu \end{aligned}$$

"Minkowski metric"

Again, to generalize to any coördinates on spacetime, look at infinitesimal intervals $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$

In a general coörd system, $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$; can convert using the chain rule: $dx^\alpha = \frac{\partial x^\alpha}{\partial x^{\overline{\mu}}} dx^{\overline{\mu}} \longrightarrow g_{\overline{\mu\nu}} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x^{\overline{\mu}}} \frac{\partial x^\beta}{\partial x^{\overline{\nu}}}$

Examples of Spacetime Coördinate Systems

- Cartesian: $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$
  $g_{tt} = -1$, $g_{xx} = g_{yy} = g_{zz} = 1$

- Spherical: $ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$
  $g_{tt} = -1$, $g_{rr} = 1$, $g_{\theta\theta} = r^2$ $g_{\phi\phi} = r^2 \sin^2\theta$

- Double-null ($u = t + x$, $v = t - x$): $ds^2 = -du\, dv + dy^2 + dz^2$
  $g_{uv} = g_{vu} = -1/2$; $g_{yy} = g_{zz} = 1$ ($g_{uu} = g_{vv} = 0$)

Metric describes infinitesimal intervals & geometry of space-time

Summary

- Einstein's theory defined by principle of relativity & invariance of speed of light
- For consistency, need to replace invariance of time w/invariance of spacetime interval $(\Delta s)^2 = -c^2(\Delta t)^2 + (\Delta \ell)^2$
- Spacetime geometry defined by invariant infinitesimal interval $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$; inertial observers use Minkowski coörds & have the special form $g_{\mu\nu} = \eta_{\mu\nu}$