INTRODUCTION TO GENERAL RELATIVITY AND COSMOLOGY

Living script

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Foreword

This material was prepared by D. Puetzfeld as part of Iowa States Astro 405/505 (Fall 2004) course. Comments and suggestions are welcome! Please report any errors, typos, etc. (probably many) back to me (dpuetz@iastate.edu). You can find the material covered in the single lectures online under www.thp.uni-koeln.de/∼dp (click on teaching). There will also be some small computer algebra programs available on this site. The script only supplements the lecture, it is not a substitute for attending the lecture!

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Chapter 1

Basics: General Relativity and Cosmology

1.1 Part I - Fun with tensors

Goal: Provide appropriate formalism for a relativistic formulation of a gravity theory. Physical laws should remain form invariant in different coordinate systems.

1.1.1 Scalars, vectors & tensors

A *scalar* is any physical quantity determined by a single numerical value which is independent of the coordinate system. Examples: (i) Charge and mass of a particle in Newtonian mechanics. (ii) Charge and rest-mass of a particle in special relativity. Simplest example of a vector is a displacement between two points, say A and B. In non-Cartesian coordinates we can reach B from A, with coordinates[1] x^μ, via the infinitesimal displacement $x^\mu + dx^\mu$. The components of such a vector between the points A and B are the differentials dx^μ. This is valid in general coordinates. Starting from the infinitesimal vector AB with components dx^μ we can construct the finite vector v^μ. Consider a curve through A and B determined by n functions of the scalar parameter λ: $x^\mu = f^\mu(\lambda)$. If A and B correspond to the values λ and $\lambda + d\lambda$ the tangent vector v^μ to the curve at A has the components $v^\mu = \frac{dx^\mu}{d\lambda}$. The infinitesimal displacement dx^μ or equivalently v^μ are prototypes for what is called a *contravariant vector.* Lets work out the components of the vector AB under coordinate transformations $x^\mu \to \tilde{x}^\mu$, which are given by n equations of the form

$$\tilde{x}^\mu = f^\mu(x^\nu), \quad \mu, \nu = 1, \ldots, n. \tag{1.1}$$

From these equations we derive

$$d\tilde{x}^\mu = \sum_\nu \frac{\partial \tilde{x}^\mu}{\partial x^\nu} dx^\nu, \tag{1.2}$$

[1] In this section all greek indices run from from $1, \ldots, n$. Later on we will confine ourselves to 4-dimensional spaces.

Now lets derive the transformation law for the vector v^μ. Since λ is a scalar parameter $d\lambda$ has the same value in both coordinate systems and consequently we find

$$\tilde{v}^\mu = \sum_\nu \frac{\partial \tilde{x}^\mu}{\partial x^\nu} v^\nu. \tag{1.3}$$

This leads us to the general definition of a contravariant vector.

Contravariant vector A *contravariant vector* is a quantity a^μ with n components depending on the coordinate system in such a way that the components a^μ in the coordinate system x^μ are related to the components $\tilde{a}^\mu$ in the coordinate system $\tilde{x}^\mu$ by a relation of the form

$$\tilde{a}^\mu = \sum_v \frac{\partial \tilde{x}^\mu}{\partial x^\nu} a^\nu. \tag{1.4}$$

Now lets come to another important point. In order to have coordinates x^μ and $\tilde{x}^\mu$, which are equally acceptable, we must be able to solve equation (1.2) with respect to dx^ν. Of course this is only possible if det $\frac{\partial \tilde{x}^\mu}{\partial x^\nu} \neq 0$ or ∞. Since

$$\sum_\alpha \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial \tilde{x}^\nu} = \sum_\alpha \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial \tilde{x}^\alpha}{\partial x^\nu} = \delta^\mu_\nu = \begin{cases} 1 & \text{if} \quad \mu = \nu \\ 0 & \text{if} \quad \mu \neq \nu \end{cases} \tag{1.5}$$

we find at once a solution of (1.2) for dx^ν, namely

$$dx^v = \sum_\mu \frac{\partial x^v}{\partial \tilde{x}^\mu} d\tilde{x}^\mu. \tag{1.6}$$

The quantity δ^μ_ν in equation (1.5) is called *Kronecker symbol*.

Covariant vector The quantity b_μ with n components is called a *covariant vector*, if for any contravariant vector a^μ

$$\sum_\mu b_\mu a^\mu = \sum_\mu \tilde{b}_\mu \tilde{a}^\mu \text{ for any } x^\mu \to \tilde{x}^\mu.$$

This sum is called the scalar product of the vectors a^μ and b_μ. From this definition follows that the transformation rule for b_μ is given by

$$b_\mu = \sum_\nu \frac{\partial \tilde{x}^\nu}{\partial x^\mu} \tilde{b}_\nu, \tag{1.7}$$

additionally, remember (1.5), we have

$$\tilde{b}_\mu = \sum_\nu \frac{\partial x^\nu}{\partial \tilde{x}^\mu} b_\nu. \tag{1.8}$$

Since we now know the difference between upper and lower indices we are going to use the so-called summation convention, i.e. we shall sum automatically over upper and lower indices

$$X^{\dots}_{\dots\alpha}Y^{\dots\alpha}_{\dots} \equiv \sum_{\alpha} X^{\dots}_{\dots\alpha}Y^{\dots\alpha}_{\dots}. \tag{1.9}$$

With this knowledge the definition of tensors of higher order is straightforward.

Contravariant/Covariant/Mixed tensors A *contravariant tensor* $T^{\mu_1\cdots\mu_k}$ of rank k is an object which transforms under coordinate transformations in such a way that, for arbitrary covariant vectors $c^{(i)}_\mu$ $(i = 1, \dots, k)$ the sum $T^{\mu_1\cdots\mu_k}c^1_{\mu_1}\dots c^k_{\mu_k}$ is a scalar, i.e.

$$T^{\mu_1\dots\mu_k}c^1_{\mu_1}\cdots c^k_{\mu_k} = \tilde{T}^{\mu_1\dots\mu_k}\tilde{c}^1_{\mu_1}\cdots\tilde{c}^k_{\mu_k} \text{ for any } x^\mu \to \tilde{x}^\mu. \tag{1.10}$$

Of course the definitions for a covariant and mixed tensor are straightforward. In the *covariant* case we have

$$T_{\mu_1\dots\mu_k}c^{\mu_1}_1\cdots c^{\mu_k}_k = \tilde{T}_{\mu_1\dots\mu_k}\tilde{c}^{\mu_1}_1\cdots\tilde{c}^{\mu_k}_k \quad \text{for any } x^\mu \to \tilde{x}^\mu, \tag{1.11}$$

and in the $k = m + l$ *mixed* case

$$T^{\mu_1\dots\mu_m}{}_{\mu_{m+1}\dots\mu_{l+m}}c^1_{\mu_1}\cdots c^m_{\mu_m}c^{\mu_{m+1}}_1\cdots c^{\mu_{l+m}}_l = \tilde{T}^{\mu_1\dots\mu_m}{}_{\mu_{m+1}\dots\mu_{l+m}}\tilde{c}^1_{\mu_1}\cdots\tilde{c}^m_{\mu_m}\tilde{c}^{\mu_{m+1}}_1\cdots\tilde{c}^{\mu_{l+m}}_l \tag{1.12}$$

for any $x^\mu \to \tilde{x}^\mu$. Hence a tensor of rank $(0, 0)$ is a scalar, $(1, 0)$ a contravariant vector, and $(0, 1)$ a covariant vector. Example: Transformation properties of a tensor[2] of rank

[2] Physical example for a tensor of rank 2 is the totally antisymmetric tensor $F^{\alpha\beta}$ of the electromagnetic field:

$$F^{\alpha\beta} := \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}.$$

The current four vector $j^\alpha := (\rho,\ j^a)$. Exercise: Show that (i) $F^{(\alpha\beta)} = 0$. (ii) Maxwell's equations take the form:

$$\begin{aligned} \partial_\beta F^{\alpha\beta} &= j^\alpha, \\ \partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} &= 0. \end{aligned}$$

(iii) The latter of the two equations can be written as $\partial_{[\alpha}F_{\beta\gamma]} = 0$. (iv) The continuity equation takes the form $\partial_\alpha j^\alpha = 0$. (v) If we introduce the four potential $\phi^\alpha = (\phi, A^a)$ the tensor of the electromagnetic field can be written as

$$F_{\alpha\beta} = 2\partial_{[\beta}\phi_{\alpha]}.$$

(Remember: $\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla\times\mathbf{A}$). (vi) A gauge transformation of the potential $\phi_\alpha \to \tilde{\phi}_\alpha = \phi_\alpha + \partial_\alpha\psi$ with a scalar ψ does not change $F^{\alpha\beta}$.

2 (for the 3 different cases $(2,0),(0,2),(1,1)$)

$$\tilde{T}^{\alpha\beta}=\frac{\partial\tilde{x}^{\alpha}}{\partial x^{\mu}}\frac{\partial\tilde{x}^{\beta}}{\partial x^{\nu}}T^{\mu\nu},\quad \tilde{T}_{\alpha\beta}=\frac{\partial x^{\mu}}{\partial\tilde{x}^{\alpha}}\frac{\partial x^{\nu}}{\partial\tilde{x}^{\beta}}T_{\mu\nu},\quad \tilde{T}^{\alpha}{}_{\beta}=\frac{\partial\tilde{x}^{\alpha}}{\partial x^{\mu}}\frac{\partial x^{\nu}}{\partial\tilde{x}^{\beta}}T^{\mu}{}_{\nu}.$$

According to the definitions given so-far we can perform simple algebraic operations with tensors. Tensors of the same order (k,l) can be added, their *sum* being again a tensor of the same order[3]. The *product* of two tensors of the order (k,l) and $\left(\hat{k},\hat{l}\right)$ will be a tensor of order $\left(k+\hat{k},l+\hat{l}\right)$[4]. A *contraction* is possible for any tensor of the order (k,l) with $k,l>0$. Putting $\mu_i=\nu_j$ in a tensor $T^{\mu_1\ldots\check{\mu}_i\ldots\mu_k}{}_{\nu_1\ldots\check{\nu}_k\ldots\nu_l}=T^{\mu_1\ldots\alpha\ldots\mu_k}{}_{\nu_1\ldots\alpha\ldots\nu_l}$ yields a tensor of rank $(k-1,l-1)$. Example: Contraction of a tensor of the order $(2,1)$, i.e. $T^{\alpha\beta}{}_{\mu}$, yields a tensor of rank $(1,0): T^{\alpha}:=T^{\alpha\beta}{}_{\beta}$[5]. The scalar $T^{\alpha}{}_{\alpha}$ is called the *trace* of a mixed tensor $T^{\alpha}{}_{\beta}$. Sometimes it is useful to split up tensors in the symmetric and antisymmetric part. The *symmetric* and *antisymmetric* part of a tensor of rank $(0,2)$ is defined by

$$T_{(\alpha\beta)}:=\frac{1}{2}\left(T_{\alpha\beta}+T_{\beta\alpha}\right),\quad T_{[\alpha\beta]}:=\frac{1}{2}\left(T_{\alpha\beta}-T_{\beta\alpha}\right).$$

The (anti)symmetry property of a tensor will be conserved in all frames[6]. We call a tensor of rank $(0,2)$ totally symmetric (antisymmetric) if $T_{\alpha\beta}=T_{(\alpha\beta)}$ $\left(T_{\alpha\beta}=T_{[\alpha\beta]}\right)$. The same statements are of course true for a tensor of rank $(2,0)$. Warning: No symmetry properties can be defined for a mixed tensor $T^{\alpha}{}_{\beta}$.The matrix of the components of $T^{\alpha}{}_{\beta}$ would be symmetric in some frame x^{μ}, but this property would not be conserved in a coordinate transformation[7]. Of course it is possible to generalize the concept of (anti)symmetry to tensors with higher rank. Example: We will call a tensor of rank $(0,3)$ (anti)symmetric if

$$\begin{aligned}T^{\alpha\beta\mu} &= T^{\beta\alpha\mu}=T^{\alpha\mu\beta}=T^{\mu\beta\alpha},\\ T^{\alpha\beta\mu} &= -T^{\beta\alpha\mu}=-T^{\alpha\mu\beta}=-T^{\mu\beta\alpha}.\end{aligned}$$

Its totally (anti)symmetric part is given by

$$\begin{aligned}T_{[\alpha\beta\mu]} &= \frac{1}{6}\left(T_{\alpha\beta\mu}+T_{\mu\beta\alpha}+T_{\mu\alpha\beta}+T_{\mu\beta\alpha}+T_{\beta\alpha\mu}+T_{\alpha\mu\beta}\right),\\ T_{(\alpha\beta\mu)} &= \frac{1}{6}\left(T_{\alpha\beta\mu}+T_{\mu\beta\alpha}+T_{\mu\alpha\beta}-T_{\mu\beta\alpha}-T_{\beta\alpha\mu}-T_{\alpha\mu\beta}\right).\end{aligned}$$

[3]Exercise: Show this explicitly for two $(1,0)$ tensors, i.e. the sum of two contravariant vectors.

[4]Exercise: Show this explicitly for two $(1,0)$ tensors, i.e. the product of two contravariant vectors.

[5]Exercise: Verify that T^{α}, i.e. $T^{\alpha\beta}{}_{\beta}$, really transforms like a contravariant vector.

[6]Exercise: Show this explicitly.

[7]Exercise: Show that the number of independent components of the symmetric tensor $T_{(\alpha\beta)}$ equals $\frac{n(n+1)}{2}$ and of the antisymmetric tensor $T_{[\alpha\beta]}$ equals $\frac{n(n-1)}{2}$.

1.1.2 Covariant derivative & connection

Consider a region of space M on which some tensor $T^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_l}$ is given at each point $P(x^\nu)$ of M, i.e. $T^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_l} = T^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_l}(x^\nu)$. In this case we call $T^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_l}$ a *tensor field* on M. Now we can ask the question whether it is possible to construct new tensor fields by differentiating the given one. If we start with the simplest tensor field $\phi = \phi(x^\nu)$ of rank $(0,0)$ the derivatives of this field $\phi_{,\mu} := \frac{\partial\phi}{\partial x^\mu}$ are the components of a covariant vector as immediately becomes clear from

$$d\phi = \phi_{,\mu}\, dx^\mu.$$

(Since $d\phi$ is a scalar and dx^μ an arbitrary contravariant vector). Next consider a tensor field of rank $(0,1)$, i.e. a covariant vector a_μ. The derivative $a_{\mu,\nu} := \frac{\partial a_\mu}{\partial x^\nu}$ has the following form in some other coordinate system

$$\tilde{a}_{\mu,\nu} \equiv \frac{\partial\tilde{a}_\mu}{\partial\tilde{x}^\nu} = \frac{\partial}{\partial\tilde{x}^\nu}\left(\frac{\partial x^\alpha}{\partial\tilde{x}^\mu}a_\alpha\right) = \frac{\partial^2 x^\alpha}{\partial\tilde{x}^\mu\partial\tilde{x}^\nu}a_\alpha + \frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\nu}a_{\alpha,\beta}. \tag{1.13}$$

This is almost the coordinate transformation formula for a covariant tensor of rank $(0,2)$. The first term would vanish in case of a linear transformation, but since we are interested in arbitrary coordinates $a_{\mu,\nu}$ is not a tensor. In order to overcome the problem with the transformation rule in equation (1.13) we rewrite it with the help of (1.7), we have

$$\tilde{a}_{\mu,\nu} - \frac{\partial^2 x^\alpha}{\partial\tilde{x}^\mu\partial\tilde{x}^\nu}\frac{\partial\tilde{x}^\beta}{\partial x^\alpha}\tilde{a}_\beta = \frac{\partial x^\alpha}{\partial\tilde{x}^\mu}\frac{\partial x^\beta}{\partial\tilde{x}^\nu}a_{\alpha,\beta}. \tag{1.14}$$

Now let us assume that there is a tensor $T_{\mu\nu}$ whose components in the coordinate system x^μ are the derivatives $a_{\mu,\nu}$. Hence we can reinterpret the lhs of (1.14) as the components of $T_{\mu\nu}$ in the coordinate system $\tilde{x}^\mu$, i.e.%

$$\tilde{T}_{\mu\nu} = \tilde{a}_{\mu,\nu} - \frac{\partial^2 x^\alpha}{\partial\tilde{x}^\mu\partial\tilde{x}^\nu}\frac{\partial\tilde{x}^\beta}{\partial x^\alpha}\tilde{a}_\beta := \tilde{a}_{\mu,\nu} - \tilde{\gamma}^\beta_{\mu\nu}\tilde{a}_\beta.$$

Therefore there might be coordinate systems in which the derivatives $a_{\alpha,\beta}$ are the components of the tensor, but we have to introduce a new coordinate dependent quantity $\gamma^\beta_{\mu\nu}$ to cover also the general case. If one now assumes that $T_{\mu\nu}$ is a tensor in all coordinate systems of the form

$$T_{\mu\nu} = a_{\mu,\nu} - \Gamma^\alpha_{\mu\nu}a_\alpha \text{ in } x^\mu \text{ and } \tilde{T}_{\mu\nu} = \tilde{a}_{\mu,\nu} - \tilde{\Gamma}^\alpha_{\mu\nu}a_\alpha \text{ in } \tilde{x}^\mu, \tag{1.15}$$

and works out[8] the transformation properties of the new coordinate dependent quantity Γ, one finds

$$\tilde{\Gamma}^\alpha_{\mu\nu} = \frac{\partial\tilde{x}^\alpha}{\partial x^\beta}\frac{\partial x^\gamma}{\partial\tilde{x}^\mu}\frac{\partial x^\delta}{\partial\tilde{x}^\nu}\Gamma^\beta_{\gamma\delta} + \frac{\partial\tilde{x}^\alpha}{\partial x^\beta}\frac{\partial^2 x^\beta}{\partial\tilde{x}^\mu\partial\tilde{x}^\nu}. \tag{1.16}$$

[8]Exercise: Work out the transformation properties of the coordinate dependent quantity $\Gamma^\alpha_{\mu\nu}$.

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The quantity $\Gamma^{\alpha}_{\mu\nu}$, which in general has n^3 independent components, is called an *affine connection*[9]. The following properties of connections follow from (1.16): (i) The difference of two connections ${}^1\Gamma^{\alpha}_{\mu\nu} - {}^2\Gamma^{\alpha}_{\mu\nu}$ is a tensor of the order $(1,2)$. (ii) If $\Gamma^{\alpha}_{\mu\nu}$ is a connection then $\Gamma^{\alpha}_{\nu\mu}$ is also a connection. (iii) $\Gamma^{\alpha}_{(\mu\nu)}$ is a symmetric connection. (iv) $\Gamma^{\alpha}_{[\mu\nu]}$ is a tensor called *torsion.* (v) A general connection can be split up into a symmetric connection and a tensor $\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{(\mu\nu)} + \Gamma^{\alpha}_{[\mu\nu]}$. (vi) It is always possible to find a coordinate transformation $x^\mu \to \tilde{x}^\mu$ at a given point P such that $\tilde{\Gamma}^{\alpha}_{\mu\nu}\Big|_P = 0$. (vii) Stronger version of the last statement: Given an arbitrary curve L we can always introduce coordinates in which $\tilde{\Gamma}^{\alpha}_{\mu\nu}\Big|_L = 0$.

In General Relativity we will be concerned only with symmetric connections[10]. The tensor in (1.15) is usually called *covariant derivative* (in this case of a $(0,1)$ tensor, i.e. covariant vector field), we define[11]

$$D_\nu a_\mu \equiv a_{\mu;\nu} := a_{\mu,\nu} - \Gamma^{\alpha}_{\mu\nu} a_\alpha .$$

Rules for covariant differentiation: (i) $\phi_{;\alpha} \equiv \phi_{,\alpha}$ for a scalar ϕ. (ii) $(A^{\cdots}{}_{\dots} B^{\cdots}{}_{\dots})_{;\alpha} = A^{\cdots}{}_{\dots;\alpha} B^{\cdots}{}_{\dots} + A^{\cdots}{}_{\dots} B^{\cdots}{}_{\dots;\alpha}$. For a tensor of rank (k,l) we have[12][13]

$$\begin{aligned} T^{\mu\nu\dots}{}_{\alpha\beta\dots;\gamma} = T^{\mu\nu\dots}{}_{\alpha\beta\dots,\gamma} \quad & + \Gamma^{\mu}_{\lambda\gamma} T^{\lambda\nu\dots}{}_{\alpha\beta\dots} + \Gamma^{\nu}_{\lambda\gamma} T^{\mu\lambda\dots}{}_{\alpha\beta\dots} + \dots \\ & - \Gamma^{\lambda}_{\alpha\gamma} T^{\mu\nu\dots}{}_{\lambda\beta\dots} - \Gamma^{\lambda}_{\beta\gamma} T^{\mu\nu\dots}{}_{\alpha\lambda\dots} + \dots . \end{aligned} \tag{1.17}$$

1.1.3 Autoparallels

We learned that the connection $\Gamma^{\alpha}_{\mu\nu}$ allows us to define the covariant derivative $a_{\mu;\nu}$ of a vector which is a tensor. Thus, we can transport a vector from a point A to a point B, the vector at point B has to be considered as the equivalent to the vector at A. We will call this operation the *parallel transport* defined by a connection $\Gamma^{\alpha}_{\mu\nu}$. In general the parallel transport between two points will depend on the path taken. Now let us consider a curve L in n dimensions given by $x^\mu = f^\mu(\lambda)$, with λ being a scalar parameter. This curve shall connect the two points A and B, additionally a^μ shall be

[9]Remark: Note that (1.15) will also be a tensor in the case $\Gamma^{\alpha}_{\mu\nu} \neq \Gamma^{\alpha}_{\nu\mu}$. Quote: " [...] the essential achievement of General Relativity, namely to overcome "ridgid" space (i.e. the inertial frame), is only indirectly connected with the introduction of a Riemannian metric. The directly relevant conceptual element is the "displacement field" ($\Gamma^{\alpha}_{\beta\gamma}$), which expresses the infinitesimal displacement of vectors. It is this which replaces the parallelism of spatially arbitrarily separated vectors fixed by the inertial frame (i.e. the equality of corresponding components) by an infinitesimal operation. This makes it possible to construct tensors by differentiation and hence to dispense with the introduction of "ridgid" space (the inertial frame). In the face of this, it seems to be of secondary importance in some sense that some particular Γ field can be deduced from a Riemannian metric [...]", A. Einstein (1955), translation by F. Gronwald, D. Hartley, F.W. Hehl.

[10]Exercise: Show that in this case the connection has $n^2(n+1)/2$ independent components.

[11]Note: In case of a non-symmetric connection there is some ambiguity at this point.

[12]Exercise: Show that the Kronecker tensor is covariantly constant, i.e. $\delta^{\mu}{}_{\nu;\alpha} = 0$.

[13]Exercise: Show that $a_{[\mu;\nu]} = a_{[\mu,\nu]} - \Gamma^{\alpha}_{[\mu\nu]} a_\alpha$.

a vector given at A. If we parallel transport a^μ along L we obtain another vector at B which we call b^μ. We assume that a^μ is tangent to L in A, i.e. $\alpha^\mu = \frac{dx^\mu}{d\lambda}\big|_A$. If we transport a^μ along L to B the final result b^μ will not necessarily be tangent to the curve. The special case in which the transported vector b^μ is tangent to the curve at every point of L will be called a geodesic curve or simply *autoparallel*. The condition which characterizes a autoparallel curve is

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = f(\lambda)\frac{dx^\mu}{d\lambda}, \tag{1.18}$$

which has to be satisfied at every point of the curve[14]. Note that the solution of this equation will be completely determined by the point A and the direction of the tangent vector at A. If we introduce another curve parameter σ, reparametrize the curve with the help of this parameter $\lambda = \lambda(\sigma)$, and choose the new parameter in such a way that

$$\frac{d^2\sigma}{d\lambda^2} = f(\lambda)\frac{d\sigma}{d\lambda},$$

then the autoparallel equation reduces to

$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma} = 0,$$

and we call σ an *affine parameter* of the geodesic.

1.1.4 Curvature

From the connection we can construct another tensor which is called the *curvature tensor* of the space. One can obtain its definition by taking the antisymmetric part

[14]For those of you who do not like this notation: If we have a manifold M, we may define the parallel transport of a vector along the curve. Let $c:]a,b[\to M$ be the curve on M, its image (for simplicity) shall be covered by a single chart (U,ϕ) whose coordinate is $x = \phi(p)$. Let X be a vector field defined along $c(t)$,

$$X|_{c(t)} = X^\mu(ct)\, e_\mu|_{c(t)} = X^\mu(ct)\frac{\partial}{\partial x^\mu}\bigg|_{c(t)}.$$

If X satisfies the condition

$$\nabla_V X = 0 \text{ for any } t \in]a,b[,$$

X is said to be parallel transported along $c(t)$. Here $V = \frac{d}{dt} = \frac{dx^\mu(c(t))}{dt} e_\mu\Big|_{c(t)}$ is the tangent vector to $c(t)$. If the tangent vector itself is parallel transported along $c(t)$, i.e.%

$$\nabla_V V = 0$$

the curve is called an autoparallel. One may also call it the straightest possible line. Note that these curves are also called *geodesics* (we will reserve this term for spaces in which we are able to measure lengths).

of the second covariant derivative of a covariant vector field a_λ, i.e. $a_{\lambda;[\mu\nu]}$[15]. We shall define the *curvature tensor* of rank $(1,3)$ as follows

$$R^\rho{}_{\lambda\mu\nu} = -\Gamma^\rho_{\lambda\mu,\nu} + \Gamma^\rho_{\lambda\nu,\mu} - \Gamma^\sigma_{\lambda\mu}\Gamma^\rho_{\sigma\nu} + \Gamma^\sigma_{\lambda\nu}\Gamma^\rho_{\sigma\mu}. \tag{1.19}$$

Hence, the connection completely determines the curvature tensor. Properties: (i) $R^\rho{}_{\lambda\mu\nu} = -R^\rho{}_{\lambda\nu\mu}$ (ii) If the connection is symmetric $R^\rho{}_{[\lambda\mu\nu]} = 0$. (iii) If the connection is symmetric and $R^\rho{}_{\lambda\mu\nu}$ vanishes in a region M, then it is possible to obtain $\Gamma^\lambda_{\mu\nu} = 0$ in M by an appropriate coordinate transformation. (iv) In case $R^\rho{}_{\lambda\mu\nu}$ vanishes in a region M than the parallel transport between two points along curves which lie entirely in M is path independent.

1.1.5 Metric & Riemannian space

Up to this point our introduction was fairly general and did not allow us to measure distances in our space. We will now switch over to a *metric space*, which is a space in which it is possible to define a scalar distance for each pair of neighboring points. There are many different examples for metric spaces: (i) The Euclidean space (in Cartesian coordinates X^α) with $d\sigma^2 = (dX^1)^2 + (dX^2)^2 + (dX^3)^2$ or (ii) the Minkowski space (in an inertial frame X^α) with $ds^2 = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2$. If we introduce general coordinates x^μ by $X^\mu = f^\mu(x^\nu)$ we have $dX^\mu = \frac{\partial X^\mu}{\partial x^\alpha}dx^\alpha$. Within such coordinates the above line elements will be of the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \tag{1.20}$$

i.e. homogeneous and quadratic in the dx^μ, with some symmetric quantity $g_{\mu\nu} = g_{(\mu\nu)}$. The relation in (1.20) characterizes a Riemannian space[16]. In general the components of tensor $g_{\mu\nu}$ are arbitrarily given functions of the coordinates, and therefore it is not possible to reduce them by a coordinate transformation to the simple form as in the Euclidean or Minkowski space, both of which are special cases of Riemannian space.

[15]Exercise: Show that

$$a_{\lambda;[\mu\nu]} = R^\rho{}_{\lambda\mu\nu}a_\rho - 2\Gamma^\rho{}_{[\mu\nu]}a_{\lambda;\rho},$$

with

$$R^\rho{}_{\lambda\mu\nu} = -\Gamma^\rho_{\lambda\mu,\nu} + \Gamma^\rho_{\lambda\nu,\mu} - \Gamma^\sigma_{\lambda\mu}\Gamma^\rho_{\sigma\nu} + \Gamma^\sigma_{\lambda\nu}\Gamma^\rho_{\sigma\mu}.$$

[16]A metric is called $\left\{\begin{array}{l}\text{positive-}\\ \text{negative-}\\ \text{in-}\end{array}\right\}$definite if $\left\{\begin{array}{l}X^2>0\\ X^2<0\\ \text{else}\end{array}\right\}$ for all vectors X^μ. Of course the metric allows us to measure angles in the usual way. For two vector X^μ and Y^μ with $X^2 \neq 0$ and $Y^2 \neq 0$ we have

$$\cos(X,Y) = \frac{g_{\mu\nu}X^\mu Y^\nu}{\sqrt{|g_{\alpha\beta}X^\alpha X^\beta|}\sqrt{|g_{\lambda\sigma}Y^\lambda Y^\sigma|}}.$$

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The tensor $g_{\mu\nu}$ of rank $(0,2)$ is called *metric*. The metric tensor allows us to construct a scalar from two infinitesimal displacements dx_1^μ and dx_2^ν at the some point A, i.e. $g_{\mu\nu}dx_1^\mu dx_2^\nu$. This is a direct generalization of the *scalar product* of two vectors as we know it from Euclidean space. Note the difference to a space without metric where we could only form scalars from two vectors if one vector is contravariant and the other covariant. Our ability to form the product of two covariant or contravariant quantities allows us to define the co-/contravariant equivalent to a contra-/covariant quantity, i.e. to raise and lower the indices of quantities. Examples:

$$a_\mu = g_{\mu\nu}a^\nu, \quad T^\nu{}_\mu = g_{\mu\alpha}T^{\nu\alpha}, \quad T_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}T^{\alpha\beta}, \ldots .$$

The contravariant form of the metric is defined via $g_{\mu\alpha}g^{\nu\alpha} = \delta_\mu^\nu$. Two important theorems with respect to the metric are: (i) At a given point it is always possible to find a coordinate transformation $x^\mu \to \tilde{x}^\mu$ such that $\tilde{g}_{\mu\nu} = \left\{ \begin{array}{lll} \pm 1 & \text{for} & \mu = \nu \\ 0 & \text{for} & \mu \neq \nu \end{array} \right.$. (ii) If we consider only real transformations the number of minus and plus signs in the diagonal form of the metric does not change. Since we now know what a metric looks like we can also work out the form of a connection in a Riemannian space. Under the assumption that the parallel transport of a vector does not change its length one obtains

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}\left(g_{\beta\nu,\mu} + g_{\mu\beta,\nu} - g_{\mu\nu,\beta}\right). \tag{1.21}$$

The quantities in (1.21) are usually called *Christoffel symbols*. This form of the connection is exclusively used in Riemannian geometry and is completely determined by the metric[17]. Let us now come back to the geodesic equation and its form in a Riemannian space. Of course the connection in (1.18) is now the connection from (1.21). Since we are now able to measure distances it is rather natural to use the proper length $s = \int_A^B ds$ of a curve as parameter λ. With $\lambda = s$ we shall have $\frac{dx^\mu}{d\lambda} = \frac{dx^\mu}{ds} \equiv u^\mu$, and the tangent vector u^μ is normalized via: $g_{\mu\nu}u^\mu u^\nu = 1$. The autoparallel equation (1.18) takes the form

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = f\left(s\right)u^\mu.$$

It can be shown that s is an affine parameter, therefore we end up with

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = 0. \tag{1.22}$$

This equation is called *geodesic equation*. Furthermore one can proof that geodesics in a Riemannian space are either the curves of maximal or minimal length connecting the points A and B. Since we already know the general definition of the curvature tensor we can now define the *Riemann tensor* which is nothing else than (1.19) together with the symmetric connection from (1.21). The following symmetries hold: $R^\rho{}_{\lambda\mu\nu} = -R^\rho{}_{\lambda\nu\mu}$,

[17]Exercise: Show that $g_{\mu\nu;\sigma} = 0$.

$R^\rho{}_{[\lambda\mu\nu]} = 0$, $R^\sigma{}_{\mu[\nu\alpha;\beta]} = 0$[18], and[19] $R_{\mu\nu\alpha\beta} = -R_{\nu\mu\alpha\beta} = R_{\alpha\beta\mu\nu}$. Thus, in a 4 dimensional space the Riemann tensor has 20 independent components. The contraction $R^\alpha{}_{\alpha\mu\nu}$ vanishes identically. The only non-vanishing contraction is the *Ricci tensor*[20]

$$R_{\mu\nu} := R^\alpha{}_{\mu\nu\alpha},$$

which (in a 4 dimensional spacetime) has 10 independent components. The contraction of the Ricci tensor

$$R := R^\mu{}_\mu$$

is called the *Ricci scalar*, the combination of the Ricci tensor and scalar

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

is called the *Einstein tensor*. The covariant divergence of the Einstein tensor vanishes, i.e. $G^\mu{}_{\nu;\mu} = 0$.

Weyl-Tensor Another important quantity with respect to the classification of different spacetimes is the so-called *Weyl tensor*. We briefly mention some of its properties here. The Riemann tensor may be expressed (in 4 dimensions) by trace-free tensor quantities in the following way

$$\begin{aligned} R_{\mu\nu\alpha\beta} &= C_{\mu\nu\alpha\beta} + \frac{1}{2}\left(g_{\mu\beta}L_{\nu\alpha} + g_{\beta\alpha}L_{\mu\beta} - g_{\mu\alpha}L_{\nu\beta} - g_{\nu\beta}L_{\mu\alpha}\right) \\ &\quad + \frac{1}{12} R\left(g_{\mu\beta}g_{\nu\alpha} - g_{\mu\alpha}g_{\nu\beta}\right), \end{aligned} \tag{1.23}$$

with $L_{\mu\nu} := R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R$, which satisfies $L^\mu{}_\mu$. The tensor $C_{\mu\nu\alpha\beta}$ with $C^\alpha{}_{\mu\nu\alpha} = 0$ is defined by (1.23) and called the Weyl tensor or conformal curvature tensor (because $\tilde{C}^\mu{}_{\nu\alpha\beta} = C^\mu{}_{\nu\alpha\beta}$ under conformal transformations $\tilde{g}_{\mu\nu} = \phi g_{\mu\nu}$). It has the same (in addition to $C^\alpha{}_{\mu\nu\alpha} = 0$) symmetries as the Riemann tensor $C_{\mu\nu\alpha\beta} = -C_{\nu\mu\alpha\beta} = -C_{\mu\nu\beta\alpha} = C_{\alpha\beta\mu\nu}$, $C_{\mu[\nu\alpha\beta]} = 0$, and therefore 10 independent components.

Isometries Without giving a derivation at this point, we notice that the condition for the existence of isometric mappings is the existence of solutions X^μ of the equation

$$X_{\mu;\nu} + X_{\nu;\mu} = 0. \tag{1.24}$$

This equation is called *Killing equation* and vectors X^μ which satisfy it are called *Killing vectors*. The existence of a Killing vector expresses a certain intrinsic symmetry property of the space.

[18] This is the *Bianchi identity*.

[19] Remember that we can lower and raise the indices with the metric.

[20] Exercise: Show that the Ricci tensor is symmetric $R_{\mu\nu} = R_{\nu\mu}$.

1.2 Part II - From Newton to Einstein

Goal: Sketch ideas which led to the formulation of GR.

1.2.1 Newton's gravitational theory

Newton's theory of gravitation has been very successful, think of the detailed study of the motion of the planets, e.g. According to Newton the gravitational force between two bodies of mass m_1 and m_2 placed at the positions $\mathbf{r}_{1,2}$ is given by

$$\mathbf{F}_{21} = \frac{G m_1 m_2}{|\mathbf{r}|^2} \frac{\mathbf{r}}{|\mathbf{r}|} = -\mathbf{F}_{12},$$

here G denotes Newton's gravitational constant[21], and the vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ points from m_1 to m_2. Lets us now distinguish the mass $m_2 = M$ as a field-generating gravitational mass and $m_1 = m$ as a test mass in the field of m_2. We introduce a gravitational field describing the force per unit mass $\mathbf{f} := \frac{\mathbf{F}}{m} = \frac{GM}{|\mathbf{r}|^2} \frac{\mathbf{r}}{|\mathbf{r}|}$, such that $\mathbf{F}_{Mm} = m\mathbf{f}$. This field may be expressed by a potential $\phi = -G \frac{M}{|\mathbf{r}|}$, i.e. $\mathbf{f} = -\nabla \phi$. By assumption the mass generating the gravitational field is M, hence $\nabla \mathbf{f} = 4\pi G M \delta^3 (\mathbf{r})$. Therefore, if we replace the pointmass by a continuous matter distribution we obtain the field equation for the gravitational potential $\nabla^2 \phi = -4\pi G \rho (\mathbf{r})$. In summary the characteristic properties of Newton's gravitational theory are: (i) It is a scalar theory (i.e. has a scalar potential ϕ and therefore a scalar source of the field, the source being the mass-density of the material distribution). (ii) The field equation is a linear partial differential equation of second order. (iii) The theory uses the pre-relativistic concepts of absolute space and absolute time, the field ϕ has no dynamic properties. Consequently Newton's theory represents an action at a distance theory. Nevertheless Newtonian gravity is a successful theory on certain length and time scales, a search for a new relativistic theory of gravitation should therefore be guided by the demand for an appropriate limit in which it reduces to Newton's theory.

1.2.2 How to formulate a relativistic gravity theory

A relativistic generalization of Newtonian gravity should at least make use of the spacetime concepts which we already know from *special relativity.* In special relativity continuous matter is described by the symmetric *stress-energy-momentum tensor* $T_{\mu\nu} = T_{(\mu\nu)}$. Example: Maxwell's stress-energy-momentum tensor for the electromagnetic field[22] ${}^{\rm Max}T_{\mu\nu}$, which has the general structure[23]

$${}^{\rm Max}T_{\mu\nu} = \begin{pmatrix} T_{00} & T_{0a} \\ T_{a0} & T_{ab} \end{pmatrix} = \begin{pmatrix} \text{energy density} & \text{momentum density} \\ \text{energy flux density} & \text{momentum flux density} \end{pmatrix}.$$

[21] In SI units we have $G = 6.673(10) \times 10^{-11}\ \mathrm{m^3 kg^{-1} s^{-2}}$.

[22] ${}^{\rm Max}T_{\alpha\beta} = -F_{\alpha\mu} F_\beta{}^\mu + \frac{1}{4} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu}$.

[23] Latin indices shall run from $a, b = 1, \ldots, 3$. Momentum flux density $\equiv$ stress.

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We can of course go ahead and mimic this structure as source for the gravitational field. Consequently, T_{00} would represent the material energy density, T_{0a} and T_{a0} the momentum and energy flux density of matter, and T_{ab} will be the mechanical stresses acting in the continuous matter which we want to consider here. Let us now try to construct a gravity theory in which $T_{\mu\nu}$ somehow represents the source of the gravitational field. An obvious choice with respect to the structure of Newton's theory would be to construct a scalar theory of gravitational interaction from the stress-energy-momentum tensor $T_{\mu\nu}$. In fact this was done a few years before the formulation of General Relativity in Minkowski spacetime[24] by using the scalar $T = \eta^{\alpha\beta} T_{\alpha\beta}$ and the field equation for the gravitational potential given by

$$\Box\phi \equiv \partial_\alpha \eta^{\alpha\beta} \partial_\beta \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = \kappa T,$$

where κ denotes a coupling constant. This is a truly relativistic gravitational theory which reduces to Newtonian gravity in the first approximation. Unfortunately, it turned out that this theory is in disagreement with observations[25]. Since it is not possible to construct a vector from the tensor $T_{\mu\nu}$ in a straightforward way[26] it is rather natural to consider a tensorial theory as a candidate for a relativistic gravity theory. In such a theory all 10 components of the tensor $T_{\mu\nu}$ will act as sources for the gravitational field, therefore the corresponding potential should also by a symmetric tensor of rank $(0, 2)$. General Relativity will turn out to be a tensorial theory of the gravitational field, the flat Minkowski background in General Relativity will be replaced by a curved Riemannian spacetime. The metric tensor $g_{\mu\nu}$ of the Riemannian space will play the role of the gravitational potential.

The equivalence principle During the development of General Relativity the equivalence principle played an important role. The equivalence principle as formulated by Einstein is a generalization of the assumption, which was already used in Newton's gravitational theory, that the gravitational mass of body and the inertial mass of a body are the same[27]. The experimental verification[28] of the equivalence of these a priori different mass concepts is embodied in the following *principle of equivalence*[29]: Gravitational and inertial forces are completely equivalent, i.e. they are of an identical nature and consequently it is impossible to separate them by any physical experiment. This principle

[24]We will use the symbol $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ for the metric in Minkowski space.

[25]Wrong prediction for the rotation of the perihelion of mercury. No deflection of light, since a scalar field cannot be coupled reasonably to the energy-momentum tensor of the Maxwell field, which is traceless.

[26]Nevertheless there have been several attempts to formulate such a theory.

[27]$\mathbf{F} = m_{\rm inert} \ddot{\mathbf{r}} = m_{\rm grav} \mathbf{f} = \mathbf{F}_{Mm}$.

[28]$(m_{\rm grav} - m_{\rm inertl})/m_{\rm gr} < 10^{-8}$ (Eötvös 1922 et al.) $< 10^{-11}$(Dicke et al. 1964) $< 10^{-12}$ (Braginski et al. 1971). A future satellite experiment (Worden et al. 1978) is targeted to achieve $< 10^{-15} - 10^{-18}$.

[29]In the literature there is usually a distinction between the so-called weak (WEP) and strong (SEP)/Einstein (EEP) equivalence principle. By using the term WEP one ususually confines the validity of the EP to mechanical systems, therefore it is the same principle as already encountered in Newton's theory. In case of the SEP all laws of nature are subject to the EP.

has some important consequences: (i) Gravitational forces (or equivalently accelerations) have to be described in the same way as inertial forces (accelerations). In an inertial frame a particle moves on a straight line described by $\frac{d^2x^\mu}{ds^2} = 0$. We already know that in a general frame this equation will look like eq. (1.22). The second term stemming from the use of a non-inertial frame. Therefore the Christoffel symbols generally describe the inertial accelerations. The EP now tells us that the gravitational accelerations are also described by the Christoffel symbols or, more precisely, the Christoffel symbols describe the sum of the inertial and gravitational accelerations[30]. Since the Christoffel symbols are derived from the metric tensor we conclude that the metric will play the role of the gravitational potential. (ii) In the general case of gravitational accelerations we cannot make the Christoffel symbols vanish everywhere and therefore space cannot be the flat Minkowski space any longer[31]. Consequently, the gravitational field will be represented by the fact that the spacetime is curved. (iii) An immediate consequence of the last two remarks is that there are no global inertial frames in General Relativity or, stated in another way, acceleration has no longer an absolute meaning in GR. There are several physical consequences which can be deduced from the validity of the EP, e.g.: (i) Necessity of the deflection of a light ray in a gravitational field. (ii) Gravitational redshift.

1.2.3 The field equations of General Relativity

Now let us come back to the field equations of a relativistic formulation of a gravity theory. We already mentioned the energy-momentum tensor as source for the gravitational field, since the scalar potential ansatz was not successful and the construction of a vectorial theory is not straightforward, it is rather natural to consider the tensorial case[32]. Hence the field equations should be of the form $F_{\mu\nu} = \kappa T_{\mu\nu}$ (with κ being a constant which has to be specified), where we have to construct the tensor on the lhs from the gravitational potential. As we saw before the role of the gravitational potential is now played by the metric. Since we want to retain the Newtonian limit we assume that $F_{\mu\nu}$ contains only derivatives of $g_{\mu\nu}$ up to the second order. In Riemannian geometry the only tensors which can be constructed from the metric $g_{\mu\nu}$ and its first and second derivatives which is linear in the second derivatives are the Riemann tensor and its contractions and the tensor $g_{\mu\nu}$ itself. Therefore an appropriate ansatz for the lhs is $F_{\mu\nu} = AR_{\mu\nu} + Bg_{\mu\nu}R + Cg_{\mu\nu}$, where A, B, and C are some constants. The requirement that the divergence (in general coordinates) of the energy-momentum vanishes, i.e. $T^{\mu\nu}{}_{;\nu} = 0$, determines[33] the constants A and B in the ansatz for $F_{\mu\nu}$, namely $A = 1$

[30] Note that there will be no way to split this sum unambiguously into two terms representing inertial and gravitational accelerations.

[31] Note that we can in principle distinguish between an accelerated observer and the gravitational field of a point mass in a non-local experiment. In case of an experiment in a gravitational field we would look for tidal effects by comparing the trajectories of two test-masses.

[32] Note that we already learned that a frame independent form of the field equations is mandatory since there are no longer any preferred coordinates.

[33] Remember our results and the definition of $G_{\mu\nu}$ in the previous section.

and $B = \frac{1}{2}$. Renaming $C \equiv \Lambda$ for historical reasons[34] we end up with the field equations of General Relativity

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + g_{\alpha\beta}\Lambda \equiv G_{\mu\nu} + g_{\alpha\beta}\Lambda = \kappa T_{\alpha\beta}. \tag{1.25}$$

For $\Lambda = 0$ the vacuum ($T_{\mu\nu} = 0$) field equations reduce to $R_{\alpha\beta} = 0$, since $R = -\kappa T^{\alpha}{}_{\alpha} \Leftrightarrow R_{\mu\nu} = \kappa \left(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T^{\mu}{}_{\mu} \right)$[35]. The coupling constant κ in the field equations is determined by demanding that (1.25) reduces to the Poisson equation in the case of weak static gravitational fields $g_{00} \simeq \left(1 + \frac{2\phi}{c^2}\right)$ and non-relativistic matter $T_{00} \sim \rho$, i.e. $\nabla^2 g_{00} \sim \frac{8\pi G}{c^4} T_{00}$, therefore $\kappa = 8\pi G / c^4$.

1.3 Part III - The Schwarzschild solution

Goal: Discuss one exact solution in GR, and derive its physical consequences.

1.3.1 Symmetry considerations

Let us now discuss one simple exact solution of the field equations of General Relativity (1.25) without cosmological constant, i.e. $G_{\mu\nu} = \kappa T_{\mu\nu}$. The solution to be discussed was found by astronomer K. Schwarzschild (1916) only a few months after Einstein published his new gravitational theory. Although the solution is simple it describes most of the general relativistic effects in the planetary system[36]. The solution found by Schwarzschild describes the gravitational field of a spherically symmetric body. We will only be concerned with the field outside of the body ($T_{\mu\nu} = 0$), hence the field equations reduce to $R_{\mu\nu} = 0$, as shown in the last section.

1.3.2 The basic form of the line element

Starting from the spherically symmetric line element[37]

$$ds^2 = e^{\nu(t,r)} c^2 dt^2 - e^{\mu(t,r)} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right). \tag{1.26}$$

Here $\nu(t,r)$ and $\mu(t,r)$ represent arbitrary functions of the coordinates. Since we want to determine the gravitational field of a spherically symmetric body at rest we demand

[34] This term is usually called cosmological constant.

[35] Exercise: How do the vacuum field equations look like if we have a non-vanishing cosmological constant. Note that in this case there is problem with the Newtonian limit of the field equations.

[36] In fact on first sight one would not expect a simple and physically meaningful solution of the general field equations.

[37] This general form of the line element can by found by means of the Killing equation (1.24). In general a spacetime is called *spherically symmetric* if there are three linear independent spacelike (${}^{n}X^{\mu}\,{}^{n}X_{\mu} < 0,\ n = 1, \ldots, 3$) Killing vectorfields $\left({}^{1}X,\ {}^{2}X,\ {}^{3}X\right)$ with: $\left[{}^{1}X, {}^{2}X\right] = {}^{3}X$, $\left[{}^{2}X, {}^{3}X\right] = {}^{1}X$, and $\left[{}^{3}X, {}^{1}X\right] = {}^{2}X$.

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that the metric is time-independent $g_{\mu\nu,0} = 0$[38]. Therefore the functions $v = v(r)$ and $\mu = \mu(r)$ will depend on the radial coordinates only. The next step is to calculate the Christoffel symbols for the metric[39], nowadays this can be easily done by the use of a computer algebra program, we only quote the final result[40]

$$
\begin{aligned}
\Gamma^1_{00} &= \frac{1}{2} e^{\nu-\mu} \nu^{'}, \quad \Gamma^1_{11} = \frac{1}{2}\mu^{'}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta, \\
\Gamma^1_{22} &= -re^{-\mu}, \quad \Gamma^2_{12} = \frac{1}{r}, \quad \Gamma^3_{23} = \frac{\cos\theta}{\sin\theta}, \\
\Gamma^0_{10} &= \frac{1}{2}\nu^{'}, \quad \Gamma^1_{33} = -re^{-\mu}\sin^2\theta, \quad \Gamma^3_{13} = \frac{1}{r}.
\end{aligned} \tag{1.27}
$$

From the connection we can calculate the non-vanishing components of the Ricci tensor $R_{\mu\nu}$

$$
R_{00} = e^{\nu-\mu}\left\{-\frac{\nu^{''}}{2} - \frac{\nu^{'}}{r} + \frac{\nu^{'}}{4}\left(\mu^{'} - \nu^{'}\right)\right\}, \tag{1.28}
$$

$$
R_{11} = \frac{\nu^{''}}{2} - \frac{\mu^{'}}{r} + \frac{\nu^{'}}{4}\left(\nu^{'} - \mu^{'}\right), \tag{1.29}
$$

$$
R_{22} = \frac{1}{\sin^2\theta} R_{33} = e^{-\mu}\left\{1 - e^{\mu} + \frac{r}{2}\left(\nu^{'} - \mu^{'}\right)\right\}. \tag{1.30}
$$

Therefore the field equations for our spherically symmetric and stationary ansatz for the metric, remember $R_{\mu\nu} = 0$, turn out to be three equations for the two unknown functions (however there is no overdetermination because the equations are not independent, remember the identity $G^{\alpha\beta}{}_{;\beta} = 0 \Rightarrow \left(R^1{}_1 - R^0{}_0 - 2R^2{}_2\right)^{'} + \left(R^1{}_1 - R^0{}_0\right)\nu^{'} + \frac{4}{r}\left(R^1{}_1 - R^2{}_2\right)$). Subtracting (1.29) from (1.28) we find

$$
\mu^{'} + \nu^{'} = 0 \Rightarrow \mu + \nu = C_1 = \text{const}. \tag{1.31}
$$

Reinserting this into (1.30) we find $1 - e^{\mu} - r\mu^{'} = 0 \Leftrightarrow \left(re^{-\mu}\right)^{'} = 1 \Rightarrow re^{-\mu} = r + C_2$ (with $C_2 = \text{const}$). Renaming the second integration constant $C_2 := -2m$ we have $e^{-\mu} = 1 - \frac{2m}{r}$. Hence we can infer from (1.31) $e^{\nu} = e^{C_1-\mu} = e^{C_1}\left(1 - \frac{2m}{r}\right)$. Therefore the line element becomes $ds^2 = e^{C_1}\left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1}dr^2 - d\Omega^2$. Rescaling of the

[38] In this case the spacetime described by the metric has a fourth Killing vector. In general a spactime is called *stationary* if there exists one timelike ($X^{\mu} X_{\mu} > 0$) Killing vectorfield. In fact this vectorfield is also orthogonal to the hypersurface $t = \text{const}$ and therefore the corresponding spacetime is called *static*. Note that the assumption of stationarity is not necessary at this point. We could also go ahead and work out the field equations for two time dependent functions $\nu = \nu(t, r)$ and $\mu = \mu(t, r)$. The result that the corresponding spacetime is static is generally known under the name *Birkhoff theorem*, i.e. every spherically symmetric solution of the vacuum field equations is necessarily static.

[39] Exercise: Check this calculation by hand! After that use a computer algebra system to verify your result.

[40] $(\dots)^{'} := \frac{\partial}{\partial r}$

time coordinate[41] finally yields the *Schwarzschild solution*

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{1.32}$$

This solution describes the exterior gravitational field outside of a spherically symmetric body. The solution has some interesting properties: (i) It depends only on one integration constant m, the meaning of this constant still has to be determined. (ii) The solution does not depend on the detailed distribution of matter inside the body. (iii) In the coordinates used the solution (1.32) becomes singular at the radius $r = 2m$.

1.3.3 Physical consequences

Let us now sketch the derivation of the trajectories of test particles in the spacetime described by the Schwarzschild line element in equation (1.32). The geodesic equation

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0, \tag{1.33}$$

takes the form[42]

$$\begin{aligned}
\ddot{r} - \frac{1}{2}\nu'\dot{r}^2 - re^\nu\dot{\theta}^2 - re^\nu\sin^2\theta\dot{\phi}^2 + \frac{1}{2}e^{2\nu}\nu'\dot{t}^2 &= 0, \\
\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 &= 0, \\
\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + \frac{2\cos\theta}{\sin\theta}\dot{\theta}\dot{\phi} &= 0, \\
\ddot{t} + \nu'\dot{r}\dot{t} &= 0.
\end{aligned} \tag{1.34}$$

Due to the spherical symmetry of the problem it is possible to consider only geodesics which lie in one plane, an advantageous choice is $\theta = \frac{\pi}{2}$, i.e. we confine ourselves to the equatorial plane. This simplifies the set in (1.34) considerably, i.e.%

$$\begin{aligned}
\ddot{r} - \frac{1}{2}\nu'\dot{r}^2 - re^\nu\dot{\phi}^2 + \frac{1}{2}e^{2\nu}\nu'\dot{t}^2 &= 0, \\
\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} &= 0, \\
\ddot{t} + \nu'\dot{r}\dot{t} &= 0.
\end{aligned} \tag{1.35}$$

Additionally, the four components of the geodesic equation are not independent. One can show that multiplication of (1.33) with $g_{\nu\mu}\dot{x}^\mu$ yields the first integral[43]

$$\begin{aligned}
\left(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right)^{\cdot} &= 0 \Rightarrow g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \text{const} \Rightarrow^{\text{normalized}} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1 \\
&\Leftrightarrow {}^{\text{SS inserted}} e^\nu\dot{t}^2 - e^{-\nu}\dot{r}^2 - r^2\dot{\phi}^2 = 1.
\end{aligned} \tag{1.36}$$

[41]Exercise: Write down this transformation explicitly. Check if the solution really satisfies the field equations (this can be done very quickly by the use of CA).

[42]$(\ldots)^{\cdot} := \frac{d}{ds}$

[43]This integral has of course a physical interpretation if we think of a particle with velocity $u^\mu = \dot{x}^\mu$ moving on this geodesic.

Where in the last line we inserted the explicit form of the Schwarzschild line element (in order to avoid many terms we use $e^\nu = \left(1 - \frac{2m}{r}\right))$. From the last two equations in (1.35) we get the following integrals

$$r^2\dot{\phi} = a = \text{const}, \quad e^\nu \dot{t} = b = \text{const}, \tag{1.37}$$

the first of them representing the conservation of angular momentum of the particle and the second one corresponding the conservation of energy of the test particle in a time-independent field[44]. By combining the last two equations from (1.37) with the one in (1.36) and substituting back for e^ν we obtain

$$\left\{\frac{d}{d\phi}\left(\frac{1}{r}\right)\right\}^2 + \frac{1}{r^2} = \frac{b^2-1}{a^2} + \frac{2m}{a^2 r} + \frac{2m}{r^3}.$$

Comparison with Newtonian theory In order to compare this result with the Newtonian theory we differentiate this equation with respect to ϕ yielding

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{m}{a^2} + \frac{3m}{r^2}. \tag{1.38}$$

The corresponding Newtonian result for a central body with mass M is

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{GM}{a^2c^2} \tag{1.39}$$

Comparison of the results in the last two equations determines the so-far unspecified integration constant within the Schwarzschild metric which is now given by $m = \frac{GM}{c^2}$. Hence m represents the mass of the central body in some relativistic units[45]. Additionally, we recognize that there will be an relativistic correction to the Newtonian result in form of the second term on the rhs of (1.38).

Some tests There are several test connected with the Schwarzschild metric: (i) The precession of the perihelia of the orbits of the inner planets $\delta\phi = \frac{6\pi m}{r_0(1-\epsilon^2)}$ radians/revolution. Here ϵ denotes the eccentricity and r_0 the semimajor axis of the orbit. Note that the positive sign in the formula denotes a precession in the same direction of the motion of the testparticle. For Mercury one has $\delta\phi = 0.1038''$ per revolution[46]. (ii) Deflection of light $\delta = \frac{2m}{r_0}$. For a light ray gracing the outer limb the sun we have

[44] At this point one should mention that the existence of any Killing vector of a space corrensponds to a first intergral of the geodesic equation, which is of course equivalent to a conservation law for the geodesic motion of a test particle.

[45] Exercise: Convince yourself that [m]= length. Verify the numerical values $m_{\text{sun}} \approx 1.5$ km, $m_{\text{earth}} \approx 0.5$ cm.

[46] Exercise: Derive the value of $\delta\phi$ for another planet in the solar of your choice. Why is it a good idea to pick mercury.

$r_0 \approx 7 \times 10^5$ km and $m \approx 1.5$ km, hence $\delta \approx 1.75''$.[47] Note that Newtonian gravity predicts only half of this value. (iii) The time delay in the field of massive body. This test was originally performed by studying reflected radar echoes from Mercury and Venus, approximately one has $\Delta t \approx 2 \times 10^{-4}$s.

Lightcones Light will move along lightlike curves, i.e. curves whose tangent vector satisfies $u^\mu u_u = 0$. From the line element in (1.32) we can infer that (in the plane $\dot{\theta} = \dot{\phi} = 0$)

$$\left(1 - \frac{2m}{r}\right)\dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 = 0. \tag{1.40}$$

For the different regions we have[48]

$$\frac{dr}{dt} = \pm\left(1 - \frac{2m}{r}\right) = \pm \left\{ \begin{array}{ll} < 0 & \text{for } r > 2m \\ > 0 & \text{for } r < 2m \\ 0 & \text{for } r = 2m \end{array} \right| .$$

Of course this equation describes the opening angle of the light cones for outgoing (+) or ingoing (−) light signals. Observer that asymptotically we have the same behaviour as in Minkowski spacetime. Additionally there is a change in the character of the coordinates for values of r smaller/larger than the gravitational radius. If we compare the two different forms of the lightcones on either side of the surface $r = 2m$ we recognize immediately that the coordinate set does not cover the whole plane and therefore should be replaced by another system. A better coordinate system was suggested by Eddington and Finkelstein who devised the following transformations for the Schwarzschild time coordinate

$$t = \tilde{t} \pm 2m \ln\left|\frac{r}{2m} - 1\right|, \; r = \tilde{r}, \; \theta = \tilde{\theta}, \; \phi = \tilde{\phi}.$$

With these relations the line element from (1.32) becomes non-diagonal, and we obtain two (depending on the sign) metrics

$$ds^2 = \left(1 - \frac{2m}{r}\right)d\tilde{t}^2 - \left(1 + \frac{2m}{r}\right)dr^2 \pm \frac{4m}{r}d\tilde{t}dr - r^2\left(d\theta + \sin^2\theta d\phi^2\right). \tag{1.41}$$

Note that the metric coefficients remain regular at $r = 2m$ and the metric has the same asymptotic behaviour as the Schwarzschild metric. Now lets investigate again the structure of the null cones of this metric. For fixed θ and ϕ we have

$$\left(1 - \frac{2m}{r}\right)d\tilde{t}^2 - \left(1 + \frac{2m}{r}\right)dr^2 \pm \frac{4m}{r}d\tilde{t}dr = 0.$$

[47] Exercise: Calculate the value of the the maximal deflection for some other astrophysical objects (Examples: Earth, a neutron star, a supermassive black hole).

[48] Exercise: Show that one can derive $t = \pm\left(r + 2m\ln|r - 2m| + \text{const}\right)$ from equation (1.40). Plot these lines in the (r, t) plane for different values of the integration constant. What does this tell you about the coordinate time?

Selecting the $-$ sign this leads to two for the radial null directions

$$\frac{dr}{d\tilde{t}} = -1, \ \frac{dr}{d\tilde{t}} = \frac{r-2m}{r+2m}. \tag{1.42}$$

The first solution represents straight lines which cover the complete coordinate and describes radially ingoing light signals and behave completely regular at $r = 2m$. The second solution, which describes radially outgoing signals, has three distinct values, i.e. $\frac{dr}{d\tilde{t}} = \begin{cases} -1 & \text{for } r \to 0 \\ 0 & \text{for } r \to 2m \\ 1 & \text{for } r \to \infty \end{cases}$. Hence for outgoing signal the coordinate system has the same problems as the original Schwarzschild version of the metric[49]. But if we consider the orientation of the light cones described by (1.42) it becomes clear that the hypersurface at $r = 2m$ represents some kind of semipermeable membrane for photons (and therefore also for observers who move on timelike curves). The hypersurface at $r = 2m$ is called an *event horizon*, since it is not possible to send some light signal to the region $r > 2m$ as soon as one has crossed the surface at $r = 2m$[50]. The $+$ sign in (1.41) describes the time reversed case. Without going into further detail we note there exist better[51], i.e. geodesically complete, coordinates for the Schwarzschild geometry which avoid the problems encountered at $r = 2m$. Finally, we remark that there is another way to verify that the singularity at $r = 2m$ is only a coordinate singularity, namely by calculating the curvature invariant $R^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu} = \frac{48m^2}{r^6}$ which shows that the spacetime described by (1.32) exhibits only an essential singularity at $r = 0$.

Interior and other exact solutions Keep in mind that we have only discussed the exterior, i.e. vacuum solution, of the field equations (1.25) in the spherically symmetric case. Of course people also modeled the interior of a star in general relativistic setup. This is in general a very complicated problem since one has to incorporate the processes which take place in a realistic star. Nevertheless people have studied highly symmetric models and found exact solutions of the field equations of GR, one of these was given by Schwarzschild in 1916 in which he assumes that the interior of the star is made of an incompressible fluid. This is the so-called *interior* Schwarzschild solution, we will not discuss this solution here.

[49] Exercise: Show that for outgoing null-curves and $-$ sign in the metric one has $t = r + 4m \ln|2m - r| + \text{const}$. Work out and plot the parametric lines $t(r)$ for both cases ($\pm$) of the Eddington-Finkelstein metric.

[50] The surface M at $r = 2m$ parametrized by

$$f(x^\mu) = r - 2m = 0,$$

has the normal vector $n_\mu = f_{,\mu} = (0, 1, 0, 0)$ and is therefore a *null-hypersurface* or *characteristic-surface*, since $n_\mu n^\mu = 0$ on M. Exercise: Work out the contravariant components $g^{\alpha\beta}$ of the metric the Eddington-Finkelstein metric and verify the statement from above.

[51] You can find the discussion in every textbook on GR $\to$ Kruskal-Szekeres coordinates.

1.4 Part IV - Basic assumptions in cosmology

Goal: Sketch fundamental ideas which form the basis of modern cosmology.

1.4.1 Two assumptions

Without going into observational detail we mention here only two assumptions which lead us to consider models of the FLRW type: (i) a global expansion and (ii) the homogeneity and isotropy of space. The first assumption goes back to an observation of Hubble, who found, by means of measuring the redshift and the luminosity of extra-galactic nebulae, a linear relationship between the radial velocity v and the distance r ascribed to the nebulae with respect to an earthbound observer. This relationship is expressed in the famous Hubble law

$$v = H_0 \, d, \tag{1.43}$$

where H_0 denotes the so-called Hubble constant. Let us stress that this does *not* single out a preferred observer in the universe as one might intuitively guess. The velocity v in the above formula arises from the original interpretation of the observed redshift as Dopplershift. In the context of General Relativity (GR) this observation can also be interpreted as a global expansion of the spacetime. The notion *redshift*, associated with the global expansion, and the notion *Dopplershift*, associated with the peculiar motion of stars, are often used synonymously in astrophysical context. However they are completely different physical effects. In order to take care of the first assumption (i), a cosmological model should incorporate something like a global scale factor $S(t)$, which describes the size of the universe. The second assumption (ii) relies on the fact that matter seems to be distributed very homogeneously in the universe at least in a statistical manner. These two assumptions motivate our ansatz for the metric and the energy-momentum in the next section.

The Robertson-Walker metric The assumption of homogeneity and isotropy and global expansion leads to the so-called Robertson-Walker metric. Using spherical coordinates (t, r, θ, ϕ) the line element is given by

$$ds^2 = dt^2 - S^2 \left(\frac{1}{1 - kr^2} \, dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right). \tag{1.44}$$

The function $S(t)$ represents the cosmic scale factor, and k can be chosen[52] to be $+1$, -1, or 0 for spaces with constant positive, negative, or zero spatial curvature, respectively.[53]

[52] After an appropriate rescaling of the coordinates.

[53] Exercise: Show that the Roberston-Walker metric is spherically symmetric. Provide the explicit form of the three Killing VFs corresponding to the spherical symmetry.

The Energy momentum tensor of an ideal fluid As already mentioned above on large scales matter appears to be distributed in a homogeneous and isotropic way. Therefore we shall assume that the (average) matter tensor $T_{\mu\nu}$ has the simple form

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu - p g_{\mu\nu}, \tag{1.45}$$

i.e. describes a perfect fluid. The two quantities $\rho = \rho\,(t)$ and $p = p\,(t)$ correspond to the energy density and pressure measured in some frame. A particularly well adapted frame is the comoving frame. In this frame the four velocity in (1.45) is reduced to $u^\mu = (1, 0, 0, 0)$[54]. We will take this frame when we work out the field equations in the next section.

1.4.2 The Friedmann equations

From the ansatz for the metric (1.44) and the energy-momentum (1.45) in we can immediately work out the form of the field equations

$$\left(\frac{\dot{S}}{S}\right)^2 + \frac{k}{S^2} - \frac{\Lambda}{3} = \frac{\kappa}{3}\rho, \tag{1.46}$$

$$2\frac{\ddot{S}}{S} + \left(\frac{\dot{S}}{S}\right)^2 + \frac{k}{S^2} - \Lambda = -\kappa p, \tag{1.47}$$

and recover the well known form of the so-called *Friedmann* equations. Hence the field equations (1.25), note that we did not assume that the cosmological constant vanishes, turn into a set of ordinary differential equations for the scale factor $S(t)$. The functions ρ, p and the parameters k, Λ depend on the model we decide to consider. Note that ρ and p are related by an equation of state $p = p(\rho)$, $p = \frac{1}{3}\rho$ in case of a radiation-dominated universe, e.g.

Epochs During its evolution the universe goes through different epochs, that are char-

acterized by the respective equation of state. The inspection of (1.46)–(1.47) reveals that the solution for the scale factor $S(t)$ depends on the choice of this equation of state. In addition to the field equations we have one differential identity (remember the definition of the Einstein tensor), i.e.%

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{1.48}$$

Let us assume that the equation of state takes the form $p\,(t) = w\,\rho\,(t)$, with $w =$const. Using (1.44), and (1.45), equation (1.48) turns into

$$\dot{\rho}S = -3\dot{S}\,(\rho + p) \quad \overset{p=w\rho}{\Rightarrow} \quad \rho = \varkappa_w S^{-3(1+w)} \sim S^{-3(1+w)}, \tag{1.49}$$

[54]In this case the velocity shall be normalized according to $g_{\mu\nu}u^\mu u^\nu = 1$.

where $\varkappa_w$ is an integration constant. Thus, we have found a relation between the energy density and the scale factor, which depends on the constant w from the equation of state. Substituting back (1.49) into (1.46) yields

$$\left(\frac{\dot{S}}{S}\right)^2 + \frac{k}{S^2} - \frac{\lambda}{3} = \frac{\kappa}{3}\varkappa_w S^{-3(1+w)}. \tag{1.50}$$

Special case A frequently discussed case is the spatially flat one with vanishing cosmological constant. Ignoring all emerging constants in the solution for $S(t)$, equation (1.50) yields

$$\left(\frac{\dot{S}}{S}\right)^2 = \frac{\kappa}{3}\varkappa_w S^{-3(1+w)} \quad \overset{\Lambda=k=0,\, \varkappa_w=1,\, p=w\rho,\, w\neq-1}{\Longrightarrow} \quad S \sim t^{\frac{2}{3(1+w)}}. \tag{1.51}$$

Bibliography

[**Beware!**] There is an enormous amount of books on relativity and cosmology, the bibliography (which is in random order) is not intended to be complete in any sense. It should only be used as a guide, which book is suitable for you depends on your taste and learning style!

[**Relativity**]

[1] W. Rindler: *Introduction to Special relativity*, Oxford University Press, Oxford (1991)
Very clear and concise introduction to SR, suitable for self-study, good book for beginners.

[2] R.U. Sexl, H.K. Urbantke: *Relativity, Groups, Particles: Special Relativity and Relativistic Symmetry in Field and Particle Physics*, Springer, New York (2001)
Contains an introduction to SR, covers many advanced topics in SR, good for people who want to learn more about the group structure of SR, maybe not the best choice for a beginner who looks for a book for self-study.

[3] W. Rindler: *Essential relativity*, Springer, New-York (1977)
Covers SR and GR, has a basic introduction to cosmology, very clear and concise, suitable for self-study, good book for beginners.

[4] R. D'Inverno: *Introducing Einstein's Relativity*, Oxford University Press, Oxford (1992)
Very good self contained introduction to GR, many pictures, suitable for self-study, good book for beginners.

[5] R. M. Wald: *General Relativity*, University of Chicago Press, Chicago (2002)
Very good self contained introduction to GR, covers also some more advanced topics, suitable for self-study but maybe not the book every beginner wants to start with.

[6] S. Weinberg: *Gravitation and cosmology.* Wiley, New York (1972)
Introduction to many topics in GR and cosmology, covers also some more advanced topics, maybe not the book every beginner wants to start with.

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Encyclopedic book on many aspects of gravitation, covers also advanced topics, suitable for self-study of you have enough time, maybe not the book every beginner wants to start with.

[**Cosmology**]

[8] P.J.E. Peebles: *Principles of physical cosmology.* Princeton University Press, Princeton (1993)
Covers many topics of cosmology in a non-technical way, very good for beginners and for self-study.

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Covers many topics in cosmological physics, includes very short introductions to GR as well as QFT, more suitable for reference not for self-study.

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